VI.7 EoM in state-space form

$$\mathbf{x} = \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ x \\ \theta_1 \\ \theta_2 \end{bmatrix} \qquad u = F \qquad y = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

From VI.5,

$$(m_c + m_1 + m_2)\ddot{x} + m_1 l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 = -b\dot{x} + F(t)$$
$$\ddot{x} + l_1 \ddot{\theta}_1 = -g\theta_1$$
$$\ddot{x} + l_2 \ddot{\theta}_2 = -g\theta_2$$

Also, $\dot{x} = \dot{x}$, $\dot{\theta}_1 = \dot{\theta}_1$, $\dot{\theta}_2 = \dot{\theta}_2$

$$\begin{bmatrix} (m_c+m_1+m_2) & m_1 l_1 & m_2 l_2 & 0 & 0 & 0 \\ 1 & l_1 & 0 & 0 & 0 & 0 \\ 1 & 0 & l_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 0 & 0 & -g \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F(t)$$

$$\text{Let } M = \begin{bmatrix} (m_c+m_1+m_2) & m_1 l_1 & m_2 l_2 \\ 1 & l_1 & 0 \\ 1 & 0 & l_2 \end{bmatrix}$$

$$\begin{bmatrix} M & O_{3\times3} \\ O_{3\times3} & I_{3\times3} \end{bmatrix} \dot{x} = \begin{bmatrix} -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 0 & 0 & -g \\ & I_{3\times3} & & & O_{3\times3} & \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} O_{3\times3} & I_{3\times3} \end{bmatrix} x + \begin{bmatrix} O_{3\times3} \end{bmatrix} u$$

Note that

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} M^{-1} & 0 \\ 0 & I \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} M^{-1} & O_{3\times3} \\ O_{3\times3} & I_{3\times3} \end{bmatrix} \begin{bmatrix} -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 0 & 0 & -g \\ & I_{3\times3} & & & O_{3\times3} & \end{bmatrix} x + \begin{bmatrix} M^{-1} & O_{3\times3} \\ O_{3\times3} & I_{3\times3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u \quad (1)$$

$$y = \begin{bmatrix} O_{3\times3} & I_{3\times3} \end{bmatrix} x + \begin{bmatrix} O_{3\times3} \end{bmatrix} u \quad (2)$$

Using Matlab's Symbolic Toolbox,

$$M^{-1} = \begin{bmatrix} \frac{1}{m_c} & -\frac{m_1}{m_c} & -\frac{m_2}{m_c} \\ -\frac{1}{m_c l_1} & \frac{m_c+m_1}{m_c l_1} & \frac{m_2}{m_c l_1} \\ -\frac{1}{m_c l_2} & \frac{m_1}{m_c l_2} & \frac{m_c+m_2}{m_c l_2} \end{bmatrix} \quad (3)$$

State-space Form:

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

$$A = \begin{bmatrix} \frac{1}{m_c} & -\frac{m_1}{m_c} & -\frac{m_2}{m_c} & 0 & 0 & 0 \\ -\frac{1}{m_c l_1} & \frac{m_c+m_1}{m_c l_1} & \frac{m_2}{m_c l_1} & 0 & 0 & 0 \\ -\frac{1}{m_c l_2} & \frac{m_1}{m_c l_2} & \frac{m_c+m_2}{m_c l_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 0 & 0 & -g \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -\frac{b}{m_c} & 0 & 0 & 0 & \frac{m_1 g}{m_c} & \frac{m_2 g}{m_c} \\ \frac{b}{m_c l_1} & 0 & 0 & 0 & -\frac{(m_c+m_1)g}{m_c l_1} & -\frac{m_2 g}{m_c l_1} \\ \frac{b}{m_c l_2} & 0 & 0 & 0 & -\frac{m_1 g}{m_c l_2} & -\frac{(m_c+m_2)g}{m_c l_2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{1}{m_c} & -\frac{m_1}{m_c} & -\frac{m_2}{m_c} & 0 & 0 & 0 \\ -\frac{1}{m_c l_1} & \frac{m_c+m_1}{m_c l_1} & \frac{m_2}{m_c l_1} & 0 & 0 & 0 \\ -\frac{1}{m_c l_2} & \frac{m_1}{m_c l_2} & \frac{m_c+m_2}{m_c l_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{m_c} \\ -\frac{1}{m_c l_1} \\ -\frac{1}{m_c l_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing out,

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{b}{m_c} & 0 & 0 & 0 & \frac{m_1 g}{m_c} & \frac{m_2 g}{m_c} \\ \frac{b}{m_c l_1} & 0 & 0 & 0 & -\frac{(m_c+m_1)g}{m_c l_1} & -\frac{m_2 g}{m_c l_1} \\ \frac{b}{m_c l_2} & 0 & 0 & 0 & -\frac{m_1 g}{m_c l_2} & -\frac{(m_c+m_2)g}{m_c l_2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{m_c} \\ -\frac{1}{m_c l_1} \\ -\frac{1}{m_c l_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} F(t)$$

$$y = \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ x \\ \theta_1 \\ \theta_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{D} F(t)$$