V.5 Linearize ball-beam EOM

(a) EOM

$$\frac{7}{5}\ddot{z} - z\dot{\theta}^2 + g\sin\theta = 0$$

$$\left(\frac{1}{3}m_2 l^2 + m_1 z^2\right)\ddot{\theta} + 2m_1 z\dot{z}\dot{\theta} + \left(m_2\frac{l}{2} + m_1 z\right)g\cos\theta = F(t)\, l\cos\theta$$

At equilibrium,

$$\ddot{z} = \dot{z} = \ddot{\theta} = \dot{\theta} = 0$$

$$\rightarrow \quad g\sin\theta_e = 0 \quad \Rightarrow \quad \boxed{\theta_e = 0}$$

$$\left(m_2\frac{l}{2} + m_1 z_e\right)g = F_e\, l$$

$$\Rightarrow \boxed{F_e = \left(\frac{1}{2}m_2 + m_1\frac{z_e}{l}\right)g} \quad (1) \text{ choose } z_e, \text{ calculate } F_e$$

Equilibrium: $(\theta_e = 0,\ F_e,\ z_e)$ s.t. (1)

(b) $\tilde{\theta} = \theta - \theta_e, \quad \tilde{z} = z - z_e, \quad \tilde{F} = F - F_e$

Linearizing nonlinear terms:

- $$z\dot{\theta}^2 \approx z_e\dot{\theta}_e^2 + \frac{\partial}{\partial z}(z\dot{\theta}^2)\Big|_e\tilde{z} + \frac{\partial}{\partial\dot{\theta}}(z\dot{\theta}^2)\Big|_e\dot{\tilde{\theta}}$$
$$= z_e\dot{\theta}_e^2 + \dot{\theta}^2\Big|_e\tilde{z} + 2z\dot{\theta}\Big|_e\dot{\tilde{\theta}}$$
$$\underline{\underline{= 0}}$$

- $$\sin\theta \approx \cancel{\sin\theta_e}^{\,0} + \frac{\partial}{\partial\theta}\sin\theta\Big|_e\tilde{\theta} = \cos\theta_e\,\tilde{\theta} = \underline{\underline{\tilde{\theta}}}$$

- $$z^2\ddot{\theta} \approx z_e^2\cancel{\ddot{\theta}_e}^{\,0} + \frac{\partial}{\partial z}z^2\ddot{\theta}\Big|_e\tilde{z} + \frac{\partial}{\partial\ddot{\theta}}z^2\ddot{\theta}\Big|_e\ddot{\tilde{\theta}}$$
$$= 2z_e\cancel{\ddot{\theta}_e}^{\,0}\tilde{z} + z_e^2\ddot{\tilde{\theta}}$$
$$= \underline{\underline{z_e^2\ddot{\tilde{\theta}}}}$$

VI, 5 (b) cont.

$$z\dot{z}\dot{\theta} \approx \cancel{z_e}\dot{z}_e\dot{\theta}_e + \frac{\partial}{\partial z}(z\dot{z}\dot{\theta})\Big|_e \tilde{z} + \frac{\partial}{\partial \dot{z}}(z\dot{z}\dot{\theta})\Big|_e \dot{\tilde{z}} + \frac{\partial}{\partial \dot{\theta}}(z\dot{z}\dot{\theta})\Big|_e \dot{\tilde{\theta}}$$

$$= \dot{z}_e\dot{\theta}_e\tilde{z} + z_e\dot{\theta}_e\dot{\tilde{z}} + \cancel{z_e}\dot{z}_e\dot{\tilde{\theta}}$$

$$\underline{\underline{= 0}}$$

$$z\cos\theta \approx z_e\cos\theta_e + \frac{\partial}{\partial z}z\cos\theta\Big|_e \tilde{z} + \frac{\partial}{\partial \theta}z\cos\theta\Big|_e \tilde{\theta}$$

$$= z_e + \cos\theta_e\,\tilde{z} - z_e\sin\theta_e\,\tilde{\theta}$$

$$= \underline{\underline{z_e + \tilde{z}}}$$

$$F\cos\theta \approx F_e\cos\theta_e + \frac{\partial}{\partial F}F\cos\theta\Big|_e \tilde{F} + \frac{\partial}{\partial \theta}F\cos\theta\Big|_e \tilde{\theta}$$

$$\doteq F_e + \cos\theta_e\,\tilde{F} - F_e\sin\theta_e\,\tilde{\theta}$$

$$= \underline{\underline{F_e + \tilde{F}}}$$

Substituting:

$$\frac{7}{5}\ddot{\tilde{z}} + g\tilde{\theta} = 0$$

$$\left(\tfrac{1}{3}m_2 l^2 + m_1 z_e^2\right)\ddot{\tilde{\theta}} + \cancel{m_2\tfrac{l}{2}g} + m_1 g(\cancel{z_e} + \tilde{z}) = l(\cancel{F_e} + \tilde{F})$$

But $F_e l = m_2\frac{l}{2}g + m_1 g z_e$

$$\Rightarrow \boxed{\begin{aligned} &\frac{7}{5}\ddot{\tilde{z}} + g\tilde{\theta} = 0 \\ &\left(\tfrac{1}{3}m_2 l^2 + m_1 z_e^2\right)\ddot{\tilde{\theta}} + m_1 g\tilde{z} = l\tilde{F} \end{aligned}}$$